

Kinetic solution to the Mach probe problem in transversely flowing strongly magnetized plasmas

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(Received 10 June 2009; published 14 September 2009)

The kinetic equation governing a strongly magnetized transverse plasma flow past a convex ion-collecting object is solved numerically for arbitrary ion to electron temperature ratio τ . The approximation of isothermal ions adopted in a recent fluid treatment of the same plasma model [I. H. Hutchinson, Phys. Rev. Lett. **101**, 035004 (2008)] is shown to have no more than a small quantitative effect on the solution. In particular, the ion flux density to an elementary portion of the object still only depends on the local surface orientation. We rigorously show that the solution can be condensed in a single “calibration factor” M_c , function of τ only, enabling Mach probe measurements of parallel and perpendicular flows by probing flux ratios at two different angles in the plane of flow and magnetic field.

DOI: [10.1103/PhysRevE.80.036403](https://doi.org/10.1103/PhysRevE.80.036403)

PACS number(s): 52.70.-m, 52.30.-q, 52.40.Hf

I. INTRODUCTION

The development of models describing the contact between plasmas and solid objects such as electric probes [1], dust [2], or ionospheric spacecraft [3] is a problem of notorious difficulty. Surfaces behave as ion and electron sinks, inducing a localized electrostatic perturbation that needs to be self-consistently resolved with the particles’ distribution functions. Although the fast-paced development of supercomputers recently started to enable first-principles simulations of a full system under various plasma conditions [4–6], analytic or semianalytic treatments are possible in the regime of strong magnetization, where the ion motion is constrained across the field lines and the dynamics are effectively one dimensional. This situation arises quite often in experimental plasmas, for instance when considering (millimeter-sized) Mach probes in tokamak scrape-off layers (SOL) [7], and in fact most theories of magnetized probe operation rely on one-dimensional models.

The present work focuses on ion collection by such strongly magnetized probes, further assuming that the Debye length is negligible compared with other scales in the problem (probe size and ion Larmor radius), and the electrons are Boltzmann distributed. The first assumption, usually well satisfied, implies that the plasma region of interest is quasineutral and the thin sheath at the probe surface need not be resolved; the latter is valid provided the probe is negatively biased enough [1]. Because the only solution to the one-dimensional divergence-free quasineutral plasma equations is spatially nonvarying, the probe presheath in the absence of transverse flow extends along the magnetic field lines and is progressively repopulated by weak effects such as ionization, anomalous cross-field transport or convective transverse flow. A comparison between ionization and anomalous transport effects can be found in Ref. [8]; we here only consider regimes where ionization is not relevant.

Upon describing the anomalous cross-field flux as diffusive, an isothermal fluid formulation of the model can be solved [9,10], providing the theoretical calibration for a Mach probe with electrodes facing parallel and antiparallel to the field when the flow is field-aligned. This approach, heu-

ristically based on an unknown diffusion rate, proved fruitful because the ion current solution only depends on the ratio of particle to momentum diffusion rates, which was argued to be close to one [11]; the absolute value of the diffusivity only affecting the presheath length. The result is usually expressed by a calibration factor M_c , such that the ratio of upstream to downstream ion flux density to the probe for a plasma flowing at isothermal parallel Mach number M_∞ , is $R \approx \exp(M_\infty/M_c)$. For equal particle and momentum diffusivities the model yields $M_c \approx 0.41$, in agreement with laser-induced fluorescence (LIF) measurements [12] to within experimental uncertainty. The kinetic formulation of the same model [13], accounting for the ion thermal dynamics, yields similar calibration factors with slight dependence on the ion to electron temperature ratio at infinity.

In situations where the plasma has a transverse flow component M_\perp due to strong radial electric fields in tokamaks’ edge for instance, diffusion is not required and purely convective equations are more appropriate. The recently solved isothermal fluid formulation of this model [14] predicts for subsonic flows an ion flux ratio $R = \exp[(M_\infty - M_\perp \cot \eta_p)/M_c]$, where η_p is the angle of probe surface to magnetic field in the plane of flow and magnetic field [see Fig. 1]. $M_c = 1/2$ exactly as anticipated in Ref. [11] for the particular case of a semi-infinite probe, but the treatment in Ref. [14] has the remarkable property of being applicable to finite-sized probes of arbitrary convex shape.

The purpose of this publication is to solve the kinetic formulation of the same convective, strongly magnetized Mach probe model. This approach naturally provides information about the ion distribution function in the presheath and is not based on approximate fluid closures. After deriving the appropriate ion kinetic equation and discussing our solution method, we show that the findings of Ref. [14] are not a consequence of the isothermal approximation and apply for arbitrary ion to electron temperature ratios. In particular, (a) flux ratios for subsonic flows are still given by $R = \exp[(M_\infty - M_\perp \cot \eta_p)/M_c]$, where M_c varies with temperature between $1/2$ and $1/\sqrt{2\pi}$, and (b) the solution applies to arbitrary-shaped convex probes. This straightforwardly allows simple calibration of four-electrode Gundestrup-like [15] Mach probes.

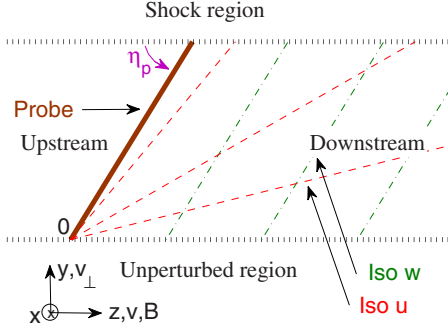


FIG. 1. (Color online) Illustration of the “planar probe” geometry. \mathbf{B} and the parallel velocity \mathbf{v} are in the \mathbf{e}_z direction, while the cross field drift \mathbf{v}_\perp is along \mathbf{e}_y . \mathbf{e}_x is the ignorable axis but supports the convective electric field. The downstream region can be parametrized by (z, y) or (u, w) , where u measures the fan angle cotangent at the origin and w measures the parallel distance to the probe.

In the entire publication, Mach numbers “ M ” are normalized to the isothermal sound speed at infinity although the ion temperature does vary.

II. QUASICOLLISIONLESS CONVECTIVE MODEL

A. Presheath equations

Let us consider a planar probe, tilted by an angle η_p in the plane of magnetic field $\mathbf{B} \parallel \mathbf{e}_z$ and ion cross-field velocity $\mathbf{v}_\perp \parallel \mathbf{e}_y$. In the limit of infinite magnetization considered here, \mathbf{v}_\perp is constant and constrained by its external driver, taken to be a uniform convective electric field in the $-\mathbf{e}_x$ direction. We further assume that the probe is negatively biased enough for the electrons to be isothermal and Boltzmann distributed [1]:

$$n_e = n_\infty \exp(\phi), \quad (1)$$

and model the plasma as quasineutral: $Zn_i = n_e = n$. Here Z is the ion charge, $n_{i,e}$ are the ion and electron densities, n_∞ is the electron density at infinity, and $\phi = eV/T_e$ is the electrostatic potential normalized to the electron temperature.

We account for anomalous cross-field transport through random ion exchange between the perturbed region (or presheath) and the outer plasma, taking place exclusively in the \mathbf{e}_x direction at a volumetric rate Ω [10]. This is admittedly an oversimplified picture but models particles and momentum diffusing into and out of the presheath at equal rate, which is consistent with reasonable physical arguments [11] as well as experiments [12]. The key requirement of the so-called “quasicollisionless” model is that Ω be much larger than the ion-electron momentum transfer Coulomb collision frequency ν_{ie} , in order for the parallel ion dynamics to be collisionless. The appropriateness of this approach, in particular with respect to a drift-diffusive parallel treatment, is discussed in Sec. II B.

The problem geometry, *a priori* two-dimensional, is shown in Fig. 1. The perturbed plasma can be divided into three distinct regions: upstream and downstream presheaths independent of each other and a shock which we do not need to analyze. In each region, we write the ion kinetic equation in steady state as

$$v \frac{\partial f}{\partial z} + v_\perp \frac{\partial f}{\partial y} - \frac{ZT_e}{m} \frac{\partial \phi}{\partial z} \frac{\partial f}{\partial v} = \Omega(f_\infty - f), \quad (2)$$

where $f(y, z, v)$ is the normalized ion distribution function in the parallel direction, m is the ion mass, and v refers to the parallel velocity variable.

In the unperturbed region, the ions are Maxwellian with drift velocity v_∞ and temperature $T_{i\infty}$. Drift velocities will usually be given in terms of isothermal Mach numbers $M_\perp = v_\perp / c_{sI}$ and $M_\infty = v_\infty / c_{sI}$, with the isothermal ion sound speed defined by

$$c_{sI} = \left(\frac{ZT_e + T_{i\infty}}{m} \right)^{1/2}. \quad (3)$$

We discuss in this publication the downstream equations, the upstream physics being recovered upon replacing (η_p, v) by $(\pi - \eta_p, -v)$. It is therefore convenient to make the change in variables illustrated in Fig. 1;

$$\begin{cases} z \\ y \end{cases} \mapsto \begin{cases} u = \frac{z}{y} \\ w = \frac{\Omega}{v_\perp} [z - y u_p] \end{cases}, \quad (4)$$

where $u = \cot \eta$ is the cotangent of the angle between the magnetic field and the position vector (fan angle) and w is a normalized distance to the probe along the parallel direction. The probe coordinates are singular at $u_p = \cot \eta_p$ and $w_p = 0$. Defining the cold-ion sound speed $c_{s0} = (ZT_e/m)^{1/2}$, Eq. (2) can then be rewritten as follows:

$$(v - v_\perp u) \frac{\partial f}{\partial u} - c_{s0}^2 \frac{\partial \phi}{\partial u} \frac{\partial f}{\partial v} = - \frac{w}{u - u_p} \left[(v - v_\perp u_p) \frac{\partial f}{\partial w} - c_{s0}^2 \frac{\partial \phi}{\partial w} \frac{\partial f}{\partial v} \right] + \frac{w}{u - u_p} v_\perp (f_\infty - f). \quad (5)$$

Equation (5) is the general formulation of the strongly magnetized Mach-probe model, including cross-field transport by both diffusion and convective motion. The relative weight of those two effects is measured by the Reynolds number,

$$\text{Re}(y) = \frac{v_\perp}{\Omega y} = \frac{u - u_p}{w}. \quad (6)$$

B. Discussion of the diffusive limit

Initial investigations of the present model by Hutchinson [9,10] in its isothermal fluid formulation and later by Chung and Hutchinson [8] in the kinetic formalism, considered parallel flows ($v_\perp = 0$) only, hence $\text{Re} = 0$ and the cross-field transport required to repopulate the probe magnetic shadow was purely diffusive. In the case $\text{Re} \ll 1$, Eq. (5) reduces to

$$(v - v_{\perp} u_p) \frac{\partial f}{\partial w} - c_{s0}^2 \frac{\partial \phi}{\partial w} \frac{\partial f}{\partial v} = v_{\perp} (f_{\infty} - f). \quad (7)$$

Van Goubergen *et al.* [16] considered nonzero convective velocity but still solved the diffusive limit implicitly assuming $\text{Re} \ll 1$ as well.

The ion distribution function at the magnetic presheath entrance (hence the collected ion current), solution of Eq. (7) at $w=0$, is clearly independent of Ω : our model is therefore not based on any estimate of this heuristic parameter. In fact Ω does not even need to be spacially uniform, rather could be function of $z - y u_p$ (parallel distance from the probe surface) provided the definition of w in Eq. (4) is replaced by $w = \Omega / v_{\perp} \int (dz - u_p dy)$. The numeric value of Ω nevertheless affects the diffusive presheath length, scaling as $\Delta w \sim c_{s1} / v_{\perp}$, i.e., $L_d \sim c_{s1} / \Omega$ in physical units.

Ω can be related to an effective transverse diffusivity D_{\perp} by $D_{\perp} \approx \Omega \Delta x^2$, where $\Delta x, \Delta y$ are the probe extents in the $\mathbf{e}_{x,y}$ directions. Let us consider a probe characterized by $\Delta x = \Delta y = 2$ mm plunged in a tokamak SOL with the following sample parameters: pure hydrogen plasma with $n_{\infty} = 10^{19} \text{ m}^{-3}$, $T_{i\infty} = T_e = 30$ eV, and $B = 5$ T. Measurements performed in the Divertor Injection Tokamak Experiment (DITE) tokamak edge [17] and in the Plasma Interaction with Surface and Components Experimental Simulator (PISCES) facility [18] show that anomalous cross-field transport in probe presheaths follows reasonably well a Bohm scaling, where Bohm diffusivity is $D_{\perp} = T_e / 16eB$. We therefore take $\Omega \approx D_{\perp} / \Delta x^2 \approx 10^5 \text{ s}^{-1}$, while the classical ion-electron Coulomb collision frequency governing parallel transport would be $\nu_{ie} \approx 1500 \text{ s}^{-1}$, i.e., $\nu_{ie} \ll \Omega$. As anticipated the parallel ion motion is collisionless, and when $\text{Re}(\Delta y) \ll 1$ Eq. (7) is the appropriate diffusive equation describing the presheath.

C. Convective limit

The question is, can we really use the diffusive equation when the cross-field velocity is not negligible? Let us consider again an equithermal plasma ($ZT_e = T_{i\infty}$), and anomalous cross-field transport described by the Bohm diffusivity $D_{\perp} = T_e / 16eB \approx \Omega \Delta x^2$. Substituting the ion isothermal sound Larmor radius $\rho_{s1} = \sqrt{(ZT_e + T_{i\infty})m} / ZeB$, the characteristic Reynolds number $\text{Re}(\Delta y)$ is

$$\text{Re}(\Delta y) = \frac{v_{\perp}}{\Omega \Delta y} \approx 32 M_{\perp} \frac{\Delta x}{\Delta y} \frac{\Delta x}{\rho_{s1}}. \quad (8)$$

The strong ion magnetization condition requires $\Delta x \gg \rho_{s1}$, let us say $\Delta x \geq 20 \rho_{s1}$ (10 Larmor diameters in Δx). If we are interested in measuring non-negligible perpendicular velocities, such as $M_{\perp} \geq 0.1$, $\text{Re}(\Delta y) \ll 1$ implies $\Delta y / \Delta x \geq 64$. Mach probes are of course not built with such a high aspect ratio, therefore Eq. (7) is only suitable to situations with $M_{\perp} \ll 1$.

For finite values of M_{\perp} , we should rather consider the opposite limit $\text{Re}(\Delta y) \gg 1$ when the second term in the right-hand side of Eq. (5) can be eliminated and the physics becomes purely convective (Ω cancels in $w \partial / \partial w$). The problem boundary conditions are that the plasma be unperturbed

when $u \rightarrow \infty$ and $w \leq w^*(u)$, where w^* is defined by $w^*(u) = (u - u_p) / \text{Re}(\Delta y)$. $w > w^*(u)$ corresponds to the shock region ($y > \Delta y$), hence not to a boundary in physical space. Provided $w \leq w^*(u)$, the above boundary conditions only depend on u ; the equation being furthermore hyperbolic in u , $\partial / \partial w = 0$ and the solutions only depend on u . This argument self-consistently holds with ϕ being a function of u only since in the quasineutral regime the potential is unambiguously determined by the local density. Of course if we were to consider a finite Debye length plasma, whose potential is governed by the three-dimensional elliptic Poisson equation, ϕ (hence f) would *a priori* depend on u , w , and presumably also the transverse position in the \mathbf{e}_x direction.

The appropriate kinetic equation that we need to solve is therefore

$$(v - v_{\perp} u) \frac{\partial f}{\partial u} - c_{s0}^2 \frac{\partial \phi}{\partial u} \frac{\partial f}{\partial v} = 0, \quad (9)$$

coupled with quasineutrality $\phi = \ln[\int f(v) dv]$. The corresponding convective presheath length scales as $L_c \sim \Delta y c_{s1} / v_{\perp}$.

D. Ion-electron symmetry

Because this work focuses on the ion saturation regime, we have so far considered the ions to be the attracted species and treated the electrons as a Boltzmann fluid. A reasonable question to ask is whether mentally inverting ions and electrons would be sufficient to study the opposite regime of electron saturation.

In the diffusive limit, the answer is most likely negative. Indeed the electron-ion Coulomb collision frequency ν_{ei} is larger than ν_{ie} by a factor equal to the ion to electron mass ratio, while anomalous cross-field transport is presumably ambipolar, i.e., Ω is equal for ions and electrons. Therefore for the tokamak edge parameters considered in Sec. II B, $\nu_{ei} \geq \Omega$ and the parallel electron motion is collisional, governed by a parallel diffusivity $D_{\parallel} = T_e / (m \nu_{ie})$.

The answer is not so definitive in the convective limit. Let us consider for example the tethered satellite system (TSS-1) flight, a low earth orbit experiment aimed at studying electron collection by a positively biased spherical subsatellite (radius $r_p = 0.8$ m) [19]. The ambient plasma conditions were $B \approx 0.3$ G, $n_{\infty} \approx 10^{11} \text{ m}^{-3}$, $v_{\perp} \approx 8$ km s^{-1} , and $T_{i\infty} \approx T_{e\infty} \approx 0.1$ eV. Therefore the electron Debye length was $\lambda_{De} \approx 7.5$ mm, and the average electron Larmor radius $\rho_e \approx 3.1$ cm, *a priori* justifying a quasineutral, strongly magnetized treatment ($\lambda_{De}, \rho_e \ll r_p$). The electron-ion Coulomb collision mean-free-path being furthermore much longer than the convective presheath length scale ($l_{ei} \sim 700$ m versus $L_c \sim 2r_p \nu_{ie} / v_{\perp} \approx 40$ m), the parallel electron motion was collisionless. Unfortunately the repelled ions being suprathermal ($v_{ti} \approx 1$ km $\text{s}^{-1} \ll v_{\perp}$) their density did not follow a Boltzmann relation, and Laframboise has shown that quasineutrality must be violated at the leading edge of the subsatellite magnetic shadow [20]. It is unclear how to adapt Eq. (9) to account for this phenomenon, and we will not attempt to do so.

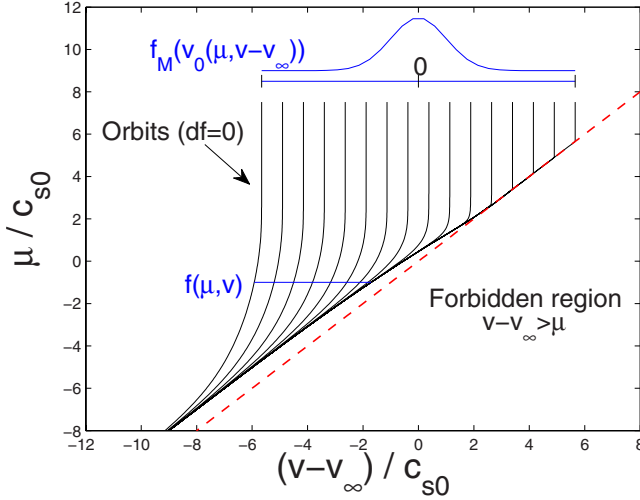


FIG. 2. (Color online) Ion orbits in μ - v space for an ion electron temperature ratio $\tau=1$ at convergence of the numerical iteration. Phase-space density at position μ [Eq. (11)] is obtained by tracing the orbit back to infinity, where the parallel ion distribution function is known to be Maxwellian with drift velocity v_∞ .

III. CONVECTIVE SOLUTION

A. Solution method

Equation (9) shows that f is conserved along (u, v) orbits that satisfy

$$\frac{dv}{du}|_{\text{Orbit}} = -c_{s0}^2 \frac{\partial \phi / \partial u}{v - v_\perp u}. \quad (10)$$

These orbits are not energy conserving but consistent with the ions only feeling the parallel gradient of the electrostatic potential while moving across the field lines. The work originating from the \mathbf{e}_y part of the potential gradient is exactly canceled by the work of the convective field $\mathbf{E}_{\text{conv}} = -\mathbf{v}_\perp \wedge \mathbf{B}$, as the ions slowly drift in the \mathbf{e}_x direction with velocity $v_x = -mc_{s0}^2 / (Ze) \nabla \phi \wedge \mathbf{B} / B^2$.

Equation (10) is invariant upon making the changes $v \mapsto v - v_\infty$ and $v_\perp u \mapsto v_\perp u - v_\infty$. We can therefore solve Eq. (9) as illustrated in Fig. 2, using the notation

$$\mu = v_\perp u - v_\infty \quad (11)$$

for compactness. We start at infinity ($\mu \gg 1$), where the normalized parallel ion distribution function is Maxwellian with drift velocity v_∞ and thermal speed $v_{ti} = (2T_{i\infty}/m)^{1/2}$, and $f_\infty(v) = f_M(v - v_\infty)$:

$$f_\infty(v) = \frac{1}{v_{ti} \sqrt{\pi}} \exp\left(-\frac{(v - v_\infty)^2}{v_{ti}^2}\right). \quad (12)$$

There a set of orbits, typically originating in the range $v_0 \in [v_\infty - 4v_{ti}, v_\infty + 4v_{ti}]$, is integrated according to Eq. (10) using an explicit fourth-order Runge-Kutta scheme. The ion distribution function at position μ and velocity v is then obtained by tracing the orbit back to its starting velocity v_0 :

$$f(\mu, v) = f_M[v_0(\mu, v - v_\infty)]. \quad (13)$$

As we do not know the potential gradient *a priori*, we start with the initial guess $\partial \phi / \partial u = M_\perp$ and iterate the orbit integration with the self-consistent potential $\phi = \ln(n)$ up to convergence, where the ion charge (electron) density is given by

$$n(\mu) = n_\infty \int f(\mu, v) dv = n_\infty \int f_M[v_0(\mu, \xi)] d\xi. \quad (14)$$

Similarly, the parallel charge flux-density in the frame moving with velocity v_∞ and ion temperature are

$$n(\mu) \langle v \rangle - v_\infty = \int \xi f_M[v_0(\mu, \xi)] d\xi, \quad (15)$$

$$T_i(\mu) = \frac{n_\infty}{n(\mu)} \int [\xi - \langle v \rangle - v_\infty]^2 f_M[v_0(\mu, \xi)] d\xi. \quad (16)$$

The main quantity of interest, the (positively defined) ion saturation flux density to the probe expressed in charge per unit time per unit surface perpendicular to the magnetic field is then given by $\Gamma_\parallel = [-n_p v(u_p) \sin \eta_p + n_p M_\perp c_{sI} \cos \eta_p] / \sin \eta_p$:

$$\Gamma_\parallel = n_p [M_\perp u_p - M_p] c_{sI}, \quad (17)$$

where $n_p = n(u_p)$ and $M_p = \langle v \rangle(u_p) / c_{sI}$. If the probe normal is in the $\{\mathbf{e}_y, \mathbf{e}_z\}$ plane (for example on a purely two-dimensional probe or on the major cross section of a sphere), the ion saturation flux density per unit probe surface is

$$\Gamma_p = \Gamma_\parallel \sin \eta_p. \quad (18)$$

B. Isothermal fluid solution

The fluid equations (continuity and momentum) equivalent to Eq. (9) are

$$\begin{aligned} \frac{1}{c_s} (\langle v \rangle - v_\perp u) \frac{\partial n}{\partial u} + \frac{n}{c_s} \frac{\partial \langle v \rangle}{\partial u} &= 0, \\ \frac{\partial n}{\partial u} + \frac{n}{c_s^2} (\langle v \rangle - v_\perp u) \frac{\partial \langle v \rangle}{\partial u} &= 0, \end{aligned} \quad (19)$$

where

$$c_s = \left(\frac{Z T_e + \gamma_i T_i}{m} \right)^{1/2} \quad (20)$$

is the Bohm ion sound speed and

$$\gamma_i = \frac{1}{T_i} \frac{dn T_i}{dn} \quad (21)$$

is the ion adiabatic index. c_s is not the speed at which sound waves would propagate in the presheath as it arises from steady-state equations, rather than the speed at which information travels in the parallel direction.

System (19) cannot be solved because it lacks closure (c_s is unknown), thus motivating our kinetic treatment. It is however clear that for the density and fluid velocity to be

nonuniform, the determinant must vanish. In other words either $n=n_\infty$ and $\langle v \rangle = v_\infty$, or $v_\perp u - \langle v \rangle = c_s$. This can be considered as the magnetized Bohm condition, valid at the probe edge regardless of the presheath model if the probe is infinite in the \mathbf{e}_y direction [21], but here derived in the convective regime for the entire plasma, without the \mathbf{e}_y -invariance requirement.

Equation (19) can be solved analytically when considering isothermal ions [14]:

$$n = n_\infty \exp(M - M_\infty), \quad (22)$$

$$M - M_\infty = \min(0, M_\perp u - M_\infty - 1), \quad (23)$$

where $M = \langle v \rangle / c_{sI}$. The isothermal approximation is exact in the limit of small ion to electron temperature ratio at infinity

$$\tau = \frac{T_{i\infty}}{ZT_e} \quad (24)$$

since the ion pressure becomes negligible compared to the electrostatic force.

C. Analogy with the plasma expansion into a vacuum

Equation (2) with $\Omega=0$ is mathematically equivalent to the one-dimensional quasineutral plasma expansion into a vacuum considered by Gurevich and Pitaevsky [Eq. (7) in Ref. [22]]

$$v \frac{\partial f}{\partial z} + \frac{\partial f}{\partial t} - \frac{ZT_e}{m} \frac{\partial \phi}{\partial z} \frac{\partial f}{\partial v} = 0, \quad (25)$$

upon replacing time t by the transverse flight time y/v_\perp . Not surprisingly therefore, the solution method described in Sec. III A essentially follows their approach. By analogy, we refer to the region $\mu \rightarrow -\infty$ as the vacuum.

An interesting point demonstrated in Ref. [22] is that in the limit $\tau \ll 1$, the ion temperature evolution is given from the isothermal solution by $T_i/T_{i\infty} = (n/n_\infty)^2$. This property has a clear physical explanation: if we assume thermal conductivity in a cold ion plasma to be negligible, f is Maxwellian at each point in space and phase-space conservation imposes invariance to $\max(f) = n / (\sqrt{2\pi}T_i)$.

D. Free-flight solution

The kinetic model [Eq. (9)] can be solved analytically in the free-flight regime when the potential gradient effects on the ion motion are neglected. The orbits in $\mu-v$ space are then vertical lines ending at $\mu = v - v_\infty$, and the ion distribution moments given by Eqs. (14)–(16) have closed form expressions. Using the notation $\mu_I = \mu / c_{sI}$ and $\omega = -c_{sI} / v_{ii} = -[(1 + 1/\tau)/2]^{1/2}$:

$$n = \frac{n_\infty}{2} \operatorname{erfc}(\omega \mu_I), \quad (26)$$

$$n \langle (v - v_\infty) \rangle = n_\infty \frac{c_{sI} / \omega}{2\sqrt{\pi}} \exp(-\omega^2 \mu_I^2), \quad (27)$$

and

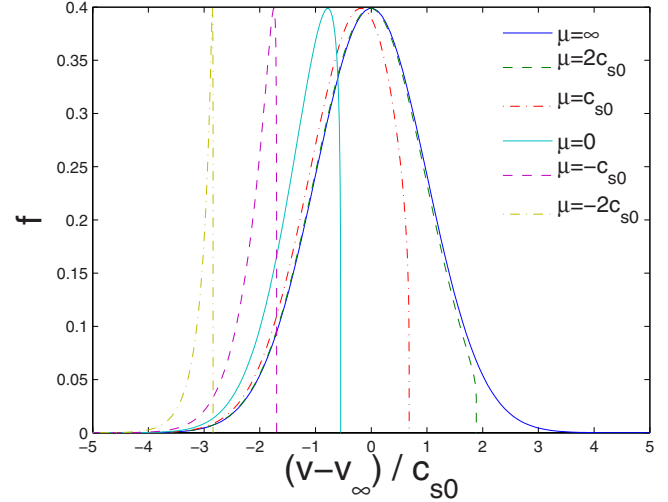


FIG. 3. (Color online) Normalized ion distribution function at different positions μ [Eq. (11)] along the presheath for originally equithermal ions and electrons ($\tau=1$).

$$\frac{T_i}{T_{i\infty}} = 1 + \frac{2\omega \mu_I \sqrt{\pi} \exp(-\omega^2 \mu_I^2) \operatorname{erfc}(\omega \mu_I) - 2 \exp(-2\omega^2 \mu_I^2)}{\pi \operatorname{erfc}(\omega \mu_I)^2}. \quad (28)$$

After tedious but straightforward algebra, the Bohm sound speed given by Eq. (20) can be calculated analytically and reduces to $c_s = v_\perp u - \langle v \rangle$. In other words, the magnetized Bohm condition discussed in Sec. III B is marginally satisfied in the entire presheath.

Free-flight calculations are justified in the limit $\tau \gg 1$ (i.e., $\omega = -1/\sqrt{2}$) since the electrostatic force becomes negligible compared to the ion pressure. We refer to this limit as the extended free-flight solution.

IV. RESULTS AND PHYSICAL DISCUSSION

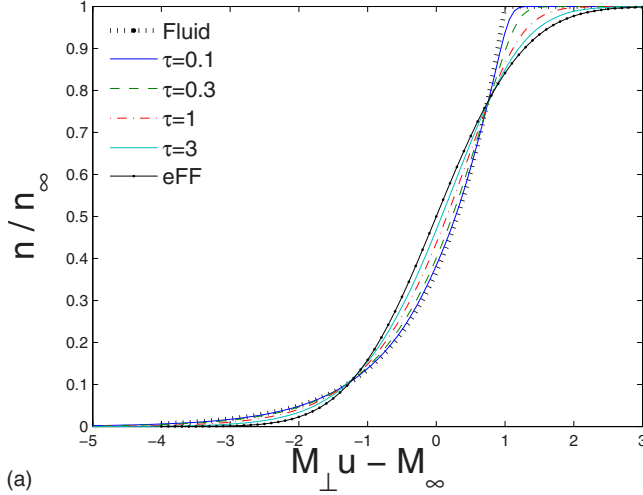
A. Plasma profiles

We start the discussion of our numerical results with the plasma profiles. Figure 3 shows the evolution of the normalized ion distribution function f with position in the presheath for originally equithermal ions and electrons ($\tau=1$). The ions cool down as they are accelerated, and f has a sharp cutoff corresponding to the probe shadowing ions streaming away from it. The sheath edge, degenerate with the probe surface in our quasineutral model, is located at $\mu = \mu_p$.

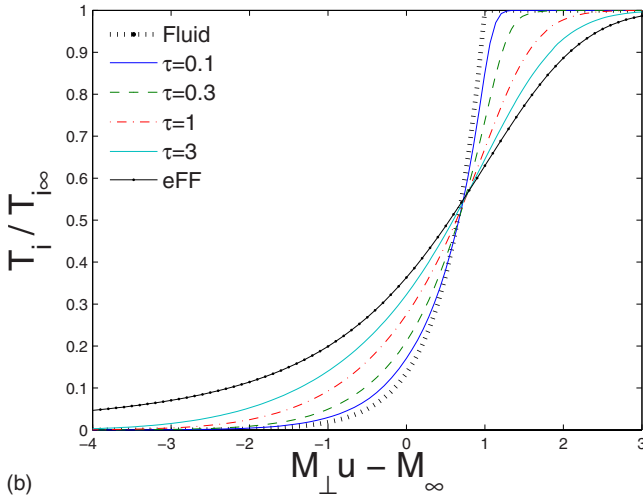
After computing the evolution of f for different temperature ratios τ , it is straightforward to take the moments [Eqs. (14)–(16)]. Density and temperature are shown in Fig. 4, with the fluid [Eq. (22) with $T_i/T_{i\infty} = (n/n_\infty)^2$] and the extended free-flight curves [Eqs. (26) and (28)].

A first noticeable result is that those analytic solutions, valid, respectively, at $\tau \ll 1$ and $\tau \gg 1$, are envelopes for the profiles at arbitrary τ , in other words the plasma properties vary monotonically with temperature ratio, which is not obvious *a priori*.

Figure 4 shows that except when $\tau=0$ and the fluid solution has a slope discontinuity at $M_\perp u - M_\infty = 1$, the tempera-



(a)



(b)

FIG. 4. (Color online) (a) Evolution of the electron density and (b) ion temperature along the presheath parametrized by $\mu_I = M_{\perp} u - M_{\infty}$ for different temperature ratios; the probe is located at $\mu_{Ip} = M_{\perp} u_p - M_{\infty}$. “Fluid” refers to Hutchinson’s fluid solution, and “eFF” to the extended free-flight solution described in Sec. III D.

ture perturbation extends much farther than the density perturbation. High-order moments are indeed more sensitive to the cutoff experienced by the ion distribution function on its positive velocity tail. Except for the singular case $\tau=0$, the ion adiabatic index [Eq. (21)] therefore goes to infinity as we approach the unperturbed plasma; this is required in order for the magnetic Bohm condition to be marginally satisfied in the entire presheath.

A further point of interest in Fig. 4(a) is that the density (hence potential) profiles are monotonic. In particular there is no localized region where the electrons are attracted, strengthening *a posteriori* our Boltzmann assumption. This is a consequence of the parallel ion motion being collisionless and the probe being at ion saturation. The situation would be fundamentally different if the probe were biased close to space potential, i.e., operating in the collisional electron collection regime yet far from electron saturation. Indeed the potential would then overshoot at approximately one electron mean-free-path from the probe sheath edge, in order for the collected electrons to overcome Coulomb friction with the

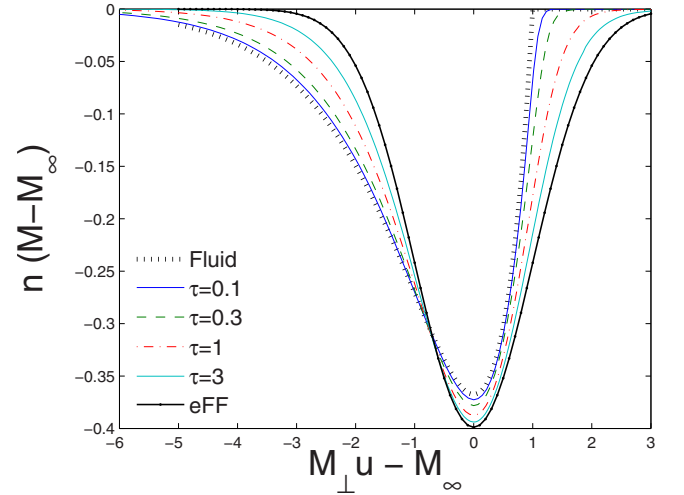


FIG. 5. (Color online) Evolution of the parallel ion flux-density in the frame moving with velocity v_{∞} [Eq. (15)], normalized to the isothermal sound speed c_{sI} . “Fluid” refers to Hutchinson’s fluid solution [Eqs. (22) and (23)], and “eFF” to the extended free-flight solution [Eq. (27)].

ions. This effect, first reported by Sanmartin in his kinetic treatment of stationary electron-collecting probes [23] and later recovered with fluid arguments [24] as well as experimentally observed [25], is absent for our purposes.

B. Ion flux density to a flat probe

The ion flux density to the probe [Eq. (17)] can be rewritten

$$\Gamma_{\parallel} = [-n_p(M_p - M_{\infty}) + n_p(M_{\perp} u_p - M_{\infty})]c_{sI}, \quad (29)$$

where $n(M - M_{\infty})c_{sI}$ corresponds to the parallel ion flux density in the frame moving with velocity v_{∞} . This term can be computed from our kinetic simulations using Eq. (15) and is plotted for different values of τ in Fig. 5.

Provided the flow Mach number is moderate and the probe surface is not grazing the magnetic field, the interesting physics lies around $\mu_I=0$, recalling the definition $\mu_I = M_{\perp} u - M_{\infty}$. It can be derived directly from the ion kinetic equation that

$$n(M - M_{\infty}) = -\frac{\Gamma_0}{c_{sI}} + O(\mu_I)^2, \quad (30)$$

$$n = n_0 + O(\mu_I), \quad (31)$$

where we defined $n_0 = n(\mu_I=0)$ and $\Gamma_0 = n_0[v_{\infty} - v(\mu_I=0)]$; recall that our calculations are performed in the downstream region of the probe, hence $n(\langle v \rangle - v_{\infty}) \leq 0$. We can therefore define α and β such that Eq. (29) expanded to third order in $\mu_{Ip} = M_{\perp} u_p - M_{\infty}$ is

$$\Gamma_{\parallel} = [\Gamma_0(1 + \alpha\mu_{Ip}^2) + n_0\mu_{Ip}(1 + \beta\mu_{Ip}^2)c_{sI}] + O(\mu_{Ip})^4. \quad (32)$$

The upstream physics is recovered upon replacing (η_p, v) by $(\pi - \eta_p, -v)$, enabling evaluation of the upstream to downstream ion current ratio $R = \Gamma_{\parallel}^{\text{Up}} / \Gamma_{\parallel}^{\text{Do}}$

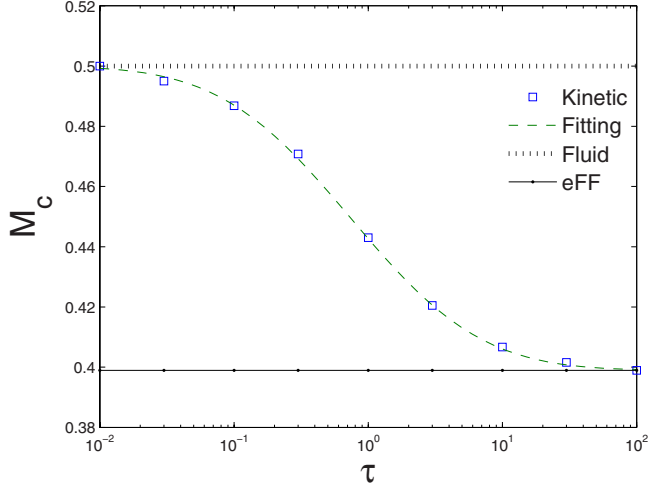


FIG. 6. (Color online) Mach probe calibration factor M_c as a function of temperature ratio τ . M_c varies from $M_c=1/2$ in cold ion plasmas (“fluid”) to $M_c=1/\sqrt{2\pi}$ in hot ion plasmas (“eFF”). “Fitting” refers to analytic expression (37).

$$R = \frac{\Gamma_0(1 + \alpha\mu_{Ip}^2) - n_0\mu_{Ip}(1 + \beta\mu_{Ip}^2)c_{sl}}{\Gamma_0(1 + \alpha\mu_{Ip}^2) + n_0\mu_{Ip}(1 + \beta\mu_{Ip}^2)c_{sl}} + O(\mu_{Ip})^4. \quad (33)$$

With the notation

$$M_c = \frac{1}{2} \frac{\Gamma_0}{n_0 c_{sl}} \quad (34)$$

and $\epsilon = 1/2 + 6(\beta - \alpha)M_c^2$, Eq. (33) simplifies to

$$R = 1 - \frac{\mu_{Ip}}{M_c} + \frac{1}{2} \frac{\mu_{Ip}^2}{M_c^2} - \frac{1 + \epsilon \mu_{Ip}^3}{6 M_c^3} + O\left(\frac{\mu_{Ip}}{M_c}\right)^4. \quad (35)$$

ϵ can be calculated numerically from our kinetic code, but this will not prove necessary as ϵ is extremely small, of the percent order. The analytic limits are $\epsilon=0$ at $\tau \ll 1$ and $\epsilon = (1 - 3/\pi)/2 \approx 0.02$ at $\tau \gg 1$.

In other words,

$$R = \frac{\Gamma_{\parallel}^{\text{Up}}}{\Gamma_{\parallel}^{\text{Do}}} = \exp\left(\frac{M_{\infty} - M_{\perp} u_p}{M_c}\right) \quad (36)$$

to second order in μ_{Ip} exactly and almost to third order, with all the physics included in M_c .

Calculation of M_c requires the temperature dependence of Γ_0 and n_0 corresponding to the slice $\mu_r = M_{\perp} u - M_{\infty} = 0$ in Figs. 4(a) and 5. Figure 6 shows our numerical solution, interpolated between the fluid and extended free-flight limits as follows:

$$M_c(\tau) = \kappa M_{c|\tau=0} + (1 - \kappa) M_{c|\tau=\infty}, \quad (37)$$

where analytic limits are

$$M_{c|\tau=0} = 1/2 \quad \text{and} \quad M_{c|\tau=\infty} = 1/\sqrt{2\pi}. \quad (38)$$

The interpolating coefficient is fitted to the numerical solution by

$$\kappa(\tau) = \frac{1}{2} \text{erfc}(0.12 + 0.40 \ln \tau). \quad (39)$$

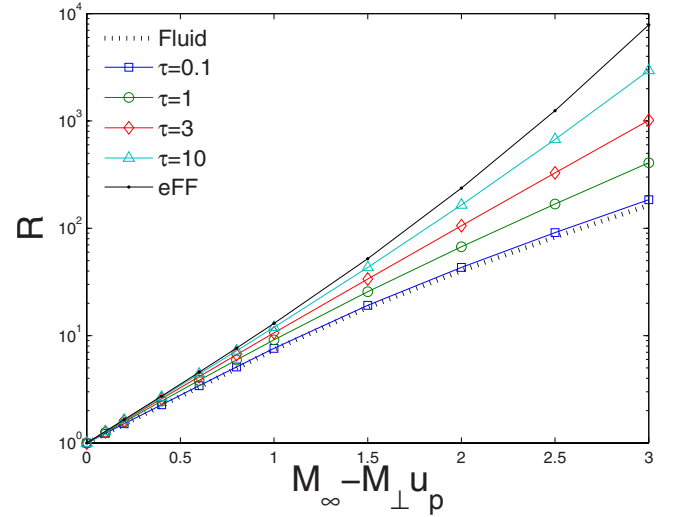


FIG. 7. (Color online) Upstream to Downstream flux ratio against $M_{\infty} - M_{\perp} u_p \in [0, 3]$ for different temperature ratios. The tangents of the flux ratio logarithms at the origin have a slope given by $1/M_c(\tau)$.

Figure 7 shows the upstream to downstream ion flux ratio against $M_{\infty} - M_{\perp} u_p \in [0, 3]$. For supersonic flows Eq. (36) is in theory not valid, the error on $\ln R$ at $M_{\infty} - M_{\perp} u_p = 2$ is however only $\sim 10\%$.

C. Extension to transverse Mach probes

The purpose of a transverse Mach probe is to measure M_{\perp} and M_{∞} . The two main competing designs are rotating probes, where a planar electrode such as schematized in Fig. 1 is rotated to measure fluxes at different tilt angles η_p , and Gundestrup-like probes, where simultaneous measurements at different angles are made by a set of electrodes spanning a single probe head [12].

Although we derived and solved our governing equations with the assumption that the probe is flat, the solution is applicable to any convex probe, upon considering η as the angle between the magnetic field and the line passing by the considered point and tangent to the probe. This configuration is illustrated in Fig. 8 for the case where the probe cross

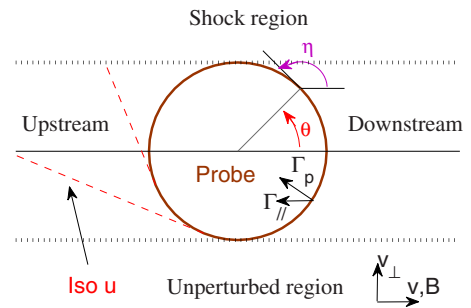


FIG. 8. (Color online) Schematic view of a convex probe with circular cross section. Each point at the probe surface is parameterized by the angle between the magnetic field and the local probe tangent η , or by $\theta = \eta - \pi/2$. The plasma solutions are invariant along the lines of constant u , the probe tangents.

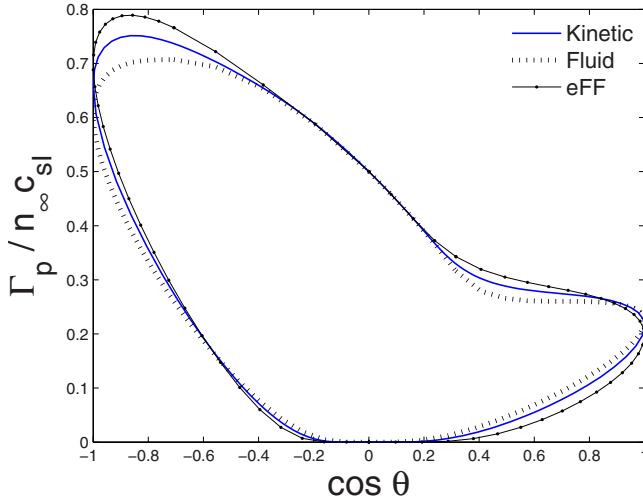


FIG. 9. (Color online) Angular distribution of ion saturation flux density [Γ_p , defined in Eq. (18)] for a drift $M_\infty=0.5$ and $M_\perp=0.5$ from our numerical kinetic solutions with $\tau=1$, compared with the isothermal fluid and extended free-flight solutions. θ is the angle between the magnetic field and the normal to the probe surface in the plane of flow and magnetic field.

section is circular. It is here easier to think in terms of $\theta = \eta - \pi/2$, angle between the magnetic field and the normal to the probe surface, because for circular cross sections it can be interpreted as the polar angle.

This was proved in the isothermal fluid formulation [14] by analyzing the characteristics of the coupled continuity and momentum equations. In the same publication, a second proof was given by considering the convex envelope of an arbitrarily shaped two-dimensional probe as the limiting case of a multifaceted polygonal surface. For this second argument to be valid here, one needs to show that information cannot propagate in the direction of decreasing u . Mathematically, this simply derives from the kinetic Eq. (9) being hyperbolic in u in the quasineutral regime considered here. The physical interpretation is that (a) the orbits shown in Fig. 2 are never reflected, in other words the ion trajectories curve toward the probe, and (b) the magnetic Bohm condition is always marginally satisfied, hence information traveling at the Bohm sound speed (in the frame locally moving with the fluid at velocity $\langle v \rangle \mathbf{e}_z + v_\perp \mathbf{e}_y$) is confined to the lines of constant u .

Figure 9 shows the angular distribution of ion saturation flux density for a drift $M_\infty=0.5$ and $M_\perp=0.5$, calculated from the kinetic equation with $\tau=1$. Comparison with the isothermal fluid and extended free-flight solutions shows that the ion temperature has little quantitative impact on the flux distribution when normalized to the isothermal sound speed. The difference is maximal at $\cos \theta = \pm 1$ and vanishes at $\cos \theta = 0$ where the probe either collects the unperturbed flow ($\theta = -\pi/2$) or zero flux ($\theta = \pi/2$).

D. Mach probe calibration

The simplest experimental procedure to find M_\perp and M_∞ is to measure the upstream to downstream flux ratio at two

different angles with either a flat or a convex Gundestrup probe: $R_1 = \Gamma_\parallel(\eta_1 + \pi) / \Gamma_\parallel(\eta_1)$ and $R_2 = \Gamma_\parallel(\eta_2 + \pi) / \Gamma_\parallel(\eta_2)$. It is desirable to avoid grazing angles with the magnetic field in order for the exponential calibration introduced in Sec. IV B to be applicable, while maximizing the tilt spacings to limit experimental noise. The optimal choice is therefore $\eta_1 = 3\pi/4$ and $\eta_2 = \pi/4$, yielding

$$M_\perp = \frac{M_c}{2} (\ln R_1 - \ln R_2), \quad (40)$$

$$M_\infty = \frac{M_c}{2} (\ln R_1 + \ln R_2). \quad (41)$$

Equations (40) and (41) require four measurements, while physically only three single measurements should be needed to find the problem's three unknowns (n_∞ , M_\perp , and M_∞). The temperature ratio τ is indeed treated as an input, supposed to be known from other diagnostics. Unfortunately M_c would only provide a three-point calibration valid to first order in the flow Mach number, each additional order requiring an additional calibration factor. Only probing flux ratios at angles $\eta + \pi$ over η as in Eq. (33) takes full advantage of the symmetries in the kinetic equation solutions, yielding the compact, quasi-third-order formula (36).

If one is interested in M_∞ only, it is in theory possible to measure R on the magnetic axis (parallel Mach probe configuration), and the calibration is then $M_\infty = M_c \ln R$. We however expect the double measurement [Eqs. (40) and (41)] to be less sensitive to finite ion Larmor radius effects. Indeed the choice $\eta_1 = 3\pi/4$ and $\eta_2 = \pi/4$ has the elegant property of being meaningful to nonmagnetized Mach probes as well. Particle in cell simulations [4] show that the unmagnetized ion flux-density distribution on a spherical probe's major cross section is approximately given by $\Gamma_p \propto \exp[-K \cos(\theta - \theta_f) v_f / 2]$, where v_f is the total flow velocity, θ_f is the angle of flow with respect to the \mathbf{e}_z axis, and $K \approx 1.34 / c_{s0}$ for $\tau \lesssim 3$; the flux ratio at angle θ is therefore $R = \Gamma_p(\theta + \pi) / \Gamma_p(\theta) = \exp[K \cos(\theta - \theta_f) v_f]$. The only possible values of η such that there exists a scalar M_c such that this flux ratio can be expressed as in Eq. (36) are $\eta = \pm \pi/4$ or $\eta = \pm 3\pi/4$ [yielding $M_c = \pm \sqrt{2} / (K c_{s1})$ on the sphere major cross section].

V. SUMMARY AND CONCLUSIONS

The probe presheath solution at ion saturation developed in this publication, derived from the kinetic Eq. (9), is valid when coherent cross-field flow dominates anomalous transport, ion magnetization is strong enough for the cross field velocity to be constant, and parallel ion collisionality is negligible. Those conditions are usually well satisfied in the presheath of Mach probes plunged in tokamak SOLs, in particular in the presence of strong radial electric fields.

Our key result is that to second and almost third order in the external flow Mach number, the ion flux ratio to electrodes whose tangents are oriented at angle $\eta + \pi$ and η with respect to the magnetic field in the plane of flow and magnetic field is given by $R = \exp[(M_\infty - M_\perp \cot \eta) / M_c]$ [Eq.

(36)]. Although the model is not isothermal, Mach numbers are normalized to the isothermal ion sound speed. M_c is the Mach probe “calibration factor,” function of ion to electron temperature ratio τ only found to vary between $M_{c|\tau=0}=1/2$ and $M_{c|\tau=\infty}=1/\sqrt{2}\pi\approx 0.4$ [Eq. (37)]. As can be seen in Fig. 6, the exponential form (36) can be used for supersonic external flows as well, albeit introducing a small error, of the order $\sim 10\%$ at $M_\infty - M_\perp \cot \eta = 2$ for instance. Measuring the flux ratios at angles $3\pi/4$ and $\pi/4$ then readily gives the external Mach numbers [Eqs. (40) and (41)].

Recalling the isothermal fluid solution [14] yields $M_c = 0.5$ regardless of τ , we conclude that the isothermal approximation induces an error less than $\sim 20\%$ on M_c , which might not be detectable in today’s Mach probe measurements. Although not a proof, it is reasonable to expect the

more sophisticated isothermal calculations accounting for diamagnetic and self-consistent convective drifts of Ref. [26] to be valid within experimental accuracy as well.

The diffusive Eq. (7), appropriate when anomalous transport dominates convection, is mathematically similar to the convective Eq. (9), hence yielding similar solutions. Chung and Hutchinson [13] found $M_c = 0.44, 0.42, 0.48$, respectively, when $\tau = 0.1, 1, 2$ for Eq. (7). This very convenient observation suggests that the Mach probe calibration is not strongly dependent upon the cross-field transport regime.

ACKNOWLEDGMENT

L.P. was supported in part by NSF/DOE Grant No. DE-FG02-06ER54891.

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